§. Unramified.

Defn $X \xrightarrow{f} Y$ is called unramified if loc. f.t. & $\Omega^1_{X/Y} = 0$.

Lemma: Unramified is stable under composition & base change.

Prop: $X \xrightarrow{f} Y$ loc. f.t. TFAE:

(1) $\Omega^1_{X/Y} = 0$.

(2) $\Delta_f : X \longrightarrow X \times_Y X$ is an open immersion.

(3) $\forall\ Z \subseteq W \longrightarrow Y$, s.t. $\mathcal{I}_Z^2 = 0$, the restriction map

$$\mathrm{Map}_{Y\text{-Schm}}(W, X) \longrightarrow \mathrm{Map}_{Y\text{-Schm}}(Z, X)$$

is injective.

pf: $(3) \Rightarrow (1)$: Let

$$\begin{tikzcd} W := \underline{\mathrm{Spec}}_X(\mathcal{O}_X \oplus \Omega^1_{X/Y}) \arrow[d, shift right] \arrow[d, shift left, "{(id,\, d_{X/Y})}"] & X \arrow[l, hook'] \\ X \arrow[d] & \\ Y & \end{tikzcd}$$

$\mathcal{O}_X \xrightarrow{(id, 0)} \mathcal{O} \oplus \Omega^1$

$(\quad\quad) \xrightarrow{pr_1} \mathcal{O}_X$.

$\Rightarrow 0 = d_{X/Y} : \mathcal{O}_X \longrightarrow \Omega^1_{X/Y} \Rightarrow \Omega^1_{X/Y} = 0$.

$(1) \Rightarrow (2)$: May work locally on X & Y, so NTS:

$A \longrightarrow B$ f.t. & $I \subseteq B \otimes_A B \xrightarrow{m} B$, then

$I^2 = I \Rightarrow V(I)$ is an open.

pf: f.t. $\Rightarrow I$ is f.g.

$I^2 = I \underset{\text{Nakayama}}{\Rightarrow}$ if $I \subseteq \mathfrak{p}$, then $I = 0$ in $(B \otimes_A B)_{\mathfrak{p}}$ $\Bigg\} \Rightarrow$ If $I \subseteq \mathfrak{p}$, then $\exists\ g \notin \mathfrak{p}$ s.t. $I = 0$ in $(B \otimes_A B)[1/g]$.

So $D(g) \subseteq V(I)$ is a nbhd of $\mathfrak{p}$ $\square$.

$(2) \Rightarrow (3)$: Suppose

$$\begin{tikzcd} W \arrow[r, shift left] \arrow[r, shift right] & X \arrow[d] \\ Z \arrow[u, hook] \arrow[ur] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} Z \arrow[r, hook] \arrow[d] & W \arrow[d] \\ X \arrow[r, "\Delta_f"] & X \times_Y X \end{tikzcd}$$

$\Rightarrow$ preimage of $\Delta_f(X)$ (which is open!) contains $|Z| = |W|$ $\square$

Prop. $X \xrightarrow{f} Y$ loc. f.t. Then it's unram. $\Longleftrightarrow$ fiberwise it's so.

pf: $\Rightarrow$ clear.

$\Leftarrow$: NTS $\Omega^1_{X/Y} = 0$, may work locally on X & Y, so:

$A \longrightarrow B$ f.t., if $\forall\, \mathfrak{p} \subseteq A$, $\Omega_{B \otimes_A \kappa(\mathfrak{p}) / \kappa(\mathfrak{p})} = 0$

Then $\Omega_{B/A} \overset{?}{=} 0$.

pf: $\mathfrak{q} \subseteq B$ & $\mathfrak{p} \subseteq A$, so that $\Omega_{B_\mathfrak{q}/A_\mathfrak{p}}$ is f.g. $B_\mathfrak{q}$-mod.

& $\left(\Omega_{B_\mathfrak{q}/A_\mathfrak{p}}\right) / \mathfrak{p} B_\mathfrak{q} = \Omega_{(B \otimes_A \kappa(\mathfrak{p}))_\mathfrak{q} / \kappa(\mathfrak{p})} = 0$

Nakayama $\Rightarrow$ $\Omega_{B_\mathfrak{q}/A_\mathfrak{p}} = \left(\Omega_{B/A}\right)_\mathfrak{q} = 0$. $\square$.

Prop: k = field, $X \longrightarrow \mathrm{Spec}(k)$ unramified $\Longleftrightarrow$ $X = \coprod_{\lambda \in \Lambda} \mathrm{Spec}(k_\lambda)$

each k_λ / k fin. sep. extn.

pf. $\Leftarrow$: Exercise

$\Rightarrow$: Step 1: Show X is locally Artinian.

$k \to A$ f.t. $\Omega_{A/k} = 0$. $\forall\, \mathfrak{p} \subseteq A$,

$\Omega_{A_\mathfrak{p}/k} \twoheadrightarrow \Omega_{\kappa(\mathfrak{p})/k}$. $\dim_{\kappa(\mathfrak{p})} \geq \mathrm{tr.d.}\,(\kappa(\mathfrak{p})/k)$.

$\Rightarrow$ $\mathrm{Spec}(A) = \coprod_{\text{finite}} \mathrm{Spec}(A_i)$, each A_i is local Artin f.t. k-alg.

& $\kappa(\mathfrak{m})/k$ fin. sep.

Step 2:

$$\begin{tikzcd} k \arrow[r] \arrow[rr, bend right, "\text{fin. sep.}"'] & \underset{\text{local Artin}}{A} \arrow[r] & k' \end{tikzcd}$$

$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A k' \to \Omega_{k'/k} \to 0$.

$\square$.

§ Étale & local structure.

Defn. (1) $X \xrightarrow{f} Y$ is étale if l.f.pr. + flat + unram. (or $\Omega^1_{X/Y} = 0$)

(2) $A \longrightarrow B$ is standard étale if $B = \left(A[x]/P\right)[1/Q]$

- $P \in A[x]$ monic

- $(P, P', Q) =$ unit ideal in $A[X]$.

Lemma: $A \to B$ std. étale $\Rightarrow$ $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ ét.
Exc.

Thm. $X \xrightarrow{f} Y$ l.f.t., $(\Omega^1_f)_x = 0$ for some $x \in X$.

Then $\exists x \in U = \mathrm{Spec}(C) \subseteq X$ s.t.

$$\begin{tikzcd} U = \mathrm{Spec}(C) \arrow[d] \arrow[r, phantom, "\subseteq"] & X \arrow[d] \\ V = \mathrm{Spec}(A) \arrow[r, phantom, "\subseteq"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr] \arrow[dr, "\text{std ét}"'] & & C \\ & B \arrow[ur] & \end{tikzcd}$$

(For l.f.t. morphisms, unram. $\Longleftrightarrow$ locally given by closed imm. $\circ$ std ét.)

pf: NTS: $A \to C \ni \mathfrak{q}$ f.t. & $(\Omega_{C/A})_{\mathfrak{q}} = 0$.

$\Rightarrow \exists g \notin \mathfrak{q}$ s.t.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C[1/g] \\ B = (A[X]/P)[1/Q] \arrow[ur] & \end{tikzcd}$$

Step 1a: WMA $A \to C = A[x_1, \dots, x_m]/I$ f.pr.

$I \twoheadrightarrow \Omega_{C/A} \otimes_C \kappa(\mathfrak{q})$, fin. subset of I already $\twoheadrightarrow$.

May replace I by f.g. sub.

Step 1b: WMA $\Omega_{C/A} = 0$: by invert some $c_1 \notin \mathfrak{q}$.

b/c $\Omega_{C/A}$ f.g.

Step 1c: WMA A f. type $/\mathbb{Z}$:

$A = \operatorname{colim}_{\lambda \in \Lambda} A_\lambda$ w/ each A_λ f.t. $/\mathbb{Z}$.
Λ filtered.

$\exists \lambda_0 \in \Lambda$ & $A_{\lambda_0} \xrightarrow{\text{f.t.}} C_{\lambda_0}$ s.t.
$$\begin{tikzcd} A_{\lambda_0} \arrow[r] \arrow[d] & C_{\lambda_0} \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$ pushout.

$$0 = \Omega_{C/A} = \operatorname{colim}_{\lambda > \lambda_0} \underbrace{C_\lambda}_{A_\lambda \otimes_{A_{\lambda_0}} C_{\lambda_0}} \otimes_{C_{\lambda_0}} \underbrace{\Omega_{C_{\lambda_0}/A_{\lambda_0}}}_{\text{f.g.}} \Rightarrow \exists \lambda > \lambda_0 \text{ s.t. } \Omega_{C_\lambda/A_\lambda} = 0.$$

Now: $A \longrightarrow C$ f.t., both f.t.$/\mathbb{Z}$ ($\Rightarrow$ Noeth.) $\Big\}$ $\overset{\text{ZMT}}{\Rightarrow}$ $\exists\, g \notin \mathfrak{q}$ & finite $A \to B$ st.

$$\begin{tikzcd} B[1/b] \arrow[r, "\cong"] & C[1/g] \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

$\Omega_{C/A} = 0 \Longrightarrow$ $\mathfrak{q}$-fin.

So WMA: $A \longrightarrow C$ finite & unram. at $\mathfrak{q}$. ($\Rightarrow \mathfrak{p}C_\mathfrak{q} = \mathfrak{q}C_\mathfrak{q}$ & $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ f.sep.)

$$\begin{matrix} A & \longrightarrow & C \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathfrak{p} & & \mathfrak{q} \end{matrix}$$

Step 1d: WMA A local.

Lemma: Exc.

$$\begin{tikzcd} & A \arrow[dl, "\text{f.pr.}"'] \arrow[dr, "\text{f.t.}"] & \\ \mathfrak{p} \subseteq B \arrow[d] & & C \ni \mathfrak{q} \arrow[d] \\ \text{then } \exists\ B[1/b] \arrow[rr, two heads] \arrow[d] & & C[1/c] \arrow[d] \\ B_\mathfrak{p} \arrow[rr, two heads] & & C_\mathfrak{q} \end{tikzcd}$$

Step 2: $A \xrightarrow{\text{fin.}} C$ (Noeth.), unram. at $\mathfrak{q}$. Then

$\mathfrak{p} \subseteq A$ local, $\mathfrak{q} \subseteq C$

$f^{-1}(\mathfrak{p}) = \{\mathfrak{q}_i, \dots, \mathfrak{q}_1, \mathfrak{q}\}$ $\quad C/\mathfrak{p}C = A/\mathfrak{p} \otimes_A C \simeq \kappa(\mathfrak{q}) \times \overline{C_1} \times \dots \times \overline{C_i}$ (local Artin)

$\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})(\alpha)$, $\overline{\mathfrak{q}_1} \ \dots\ \overline{\mathfrak{q}_i}$.

Choose $c \in C$ lifting the elt $(\alpha, 0, \dots, 0)$.

$\mathrm{Im}(A[x] \longrightarrow C) =: R \ni \mathfrak{q}'$

$$\begin{tikzcd} & R \arrow[dr, "\text{fin.}"] & \\ \mathfrak{p} \subseteq A \arrow[ur, "\text{fin.}"] \arrow[rr] & & C \supseteq \mathfrak{q} \end{tikzcd}$$

Claim: $C \otimes_R R_{\mathfrak{q}'}$ has only 1 max'l ideal, $\mathfrak{q}$.

Otherwise: say $\mathfrak{q}_1$ is also $\Longrightarrow$ $x \in \mathfrak{q}_1 \cap R$, $\notin \mathfrak{q}'$ $\quad \times$.

$\Rightarrow R_{\mathfrak{q}'} \cong C_\mathfrak{q}$.

WMA

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[x] \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ \kappa(\mathfrak{p}) \arrow[r] & \kappa(\mathfrak{p})[x] \arrow[r, two heads] & C/\mathfrak{p}C = \kappa(\mathfrak{p})[x]/\overline{P}(x) \end{tikzcd}$$

monic poly. of deg n.

$\overset{\text{Nakayama}}{\Longrightarrow}$ $A \cdot \{1, x, \dots, x^{n-1}\} \twoheadrightarrow C$.

$\Longrightarrow \exists\ P(x) = x^n + \sum a_i x^i \in \ker(A[x] \twoheadrightarrow C)$.

& $\overline{P}(x) \in \kappa(\mathfrak{p})[x]$ necessarily image of P.

$$\Rightarrow \begin{tikzcd} A \arrow[r] & A[x]/P(x) \arrow[r, two heads] & C \arrow[l, bend right, "\text{isom mod } \mathfrak{p}"'] \\ & \mathfrak{q}' \arrow[u, phantom, "\subseteq" rotate=90] & \mathfrak{q} \arrow[u, phantom, "\subseteq" rotate=90] \\ & (\quad)_{\mathfrak{q}'} \arrow[r, two heads] & (\ )_{\mathfrak{q}} \end{tikzcd}$$

isom after mod $\mathfrak{p}$. $\Rightarrow P'(x) \notin \mathfrak{q}'$.

$$\Rightarrow \begin{tikzcd} & A \arrow[dl] \arrow[drr] & & \\ (A[x]/p(x))[1/p'(x)] \arrow[d] \arrow[r] & & & C \arrow[d] \arrow[dl] \\ \ & (\ )[1/\underbrace{p'Q}_{\text{new } Q}] \arrow[r, two heads] & C[1/c] & \ \\ (\quad)_{\mathfrak{q}'} \arrow[rrr, two heads] & & & C_{\mathfrak{q}} \end{tikzcd}$$

Thm: $X \xrightarrow{\text{l.f.pr.}} Y$, étale at $x \in X$ $\left(\mathcal{O}_{X,x} / \mathcal{O}_{Y,f(x)} \text{ flat } \& \ (\Omega_{X/Y})_x = 0 \right)$.

Then $\exists\, x \in U = \mathrm{Spec}(B) \subseteq X$
$$\begin{tikzcd} U = \mathrm{Spec}(B) \arrow[d] \arrow[r, phantom, "\subseteq"] & X \arrow[d] \\ V = \mathrm{Spec}(A) \arrow[r, phantom, "\subseteq"] & Y \end{tikzcd}$$
s.t. $A \to B$ std Ét.

Fact (OORC) $R \to S$ f.pr., M/S f.pr. $\{ s \in \mathrm{Spec}(S) \mid M_s \text{ flat}/R \}$ open in $\mathrm{Spec}(S)$.

pf of Thm: NTS: $A \to B$ & $\mathfrak{p} \subseteq B$ étale $\Rightarrow A \to B[1/b]$ std ét for some $b \notin \mathfrak{p}$.

By previous: $A \to (A[x]/p)[1/Q] \twoheadrightarrow B[1/b_1]$ for some $b_1 \notin \mathfrak{p}$.

by invert further, WMA fiber above $\mathfrak{p}$ is unique

Now: std Ét A-alg $R \twoheadrightarrow B$. $I := \ker$ f.g./R.
$\tilde{\mathfrak{q}} \subseteq R$, $\mathfrak{q} \subseteq B$.

$I \hookrightarrow R \twoheadrightarrow B$.

$- \otimes_A A/\mathfrak{p}A \xRightarrow{B/A \text{ flat}} 0 \to I/\mathfrak{p}I \to R/\mathfrak{p}R \xrightarrow{\cong} B/\mathfrak{p}B \to 0. \Rightarrow I = \mathfrak{p}I \Rightarrow I = \tilde{\mathfrak{q}}I$

$\Rightarrow I_{\tilde{\mathfrak{q}}} = 0 \Rightarrow$ invert some $r \notin \tilde{\mathfrak{q}}$, $I[1/r] = 0$ $\square$.